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# Comparison of the solutions of a phase-lagging heat transport equation and damped wave equation

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**Abstract**

The phase-lagging equation (PLE) is a new heat conduction equation which is different from the traditional heat equation since there exists a time lag of the heat-flux vector, while the damped wave equation (DWE) is its first-order approximation. In this article, we study the difference between the solutions of the PLE and the DWE by investigating the solutions of a test problem. Results show that the level of the solution obtained by the PLE is smaller in magnitude than the one obtained by the DWE, and that the DWE is a good approximation to the PLE when the time lag is small.


## 1. Introduction

For the problem of heat transported by conduction in which the heat pulses are transmitted by waves at finite but perhaps high speed [1,2], particularly, under low temperature or high heat-flux conditions, the lagging response must be included [1–6]. Thus, the traditional Fourier's law [7]

$$\vec{q}(\vec{r},\tau) = -K\nabla\theta(\vec{r},\tau) \tag{1}$$

should be modified as follows [8]:

$$\vec{q}(\vec{r},\tau+\lambda_0) = -K\nabla\theta(\vec{r},\tau), \tag{2}$$

where $\vec{q}$ is the heat-flux vector, $K$ is the thermal conductivity, $\theta$ is the absolute temperature, $\vec{r}$ is the position vector, and $\tau$ is the time. Here, $\lambda_0(>0)$ represents the time lag required to establish steady thermal conduction in a volume element once a temperature gradient has been imposed across it. This quantity has been experimentally determined for a number of materials [1,9,10]. Combined with the energy conservation law

$$\rho C_p \frac{\partial\theta(\vec{r},\tau)}{\partial\tau} + \nabla\cdot\vec{q}(\vec{r},\tau) = 0, \tag{3}$$

where $\rho$ is the mass density, $C_p$ is the specific heat at constant pressure, and the thermal source term was assumed to be zero for simplicity, Eq. (2) results in the following phase-lagging (i.e., delay) heat transport equation:

$$\frac{\partial\theta(\vec{r},\tau+\lambda_0)}{\partial\tau} = \kappa\nabla^2\theta(\vec{r},\tau), \tag{4}$$

where $\kappa = K/(\rho C_p)$ is the thermal diffusivity. On the other hand, approximating Eq. (2) by its first-order Taylor series expansion yields the Maxwell–Cattaneo (MC) thermal flux law [1,3,8,11,12], namely

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**Nomenclature**

| | | | |
|---|---|---|---|
| $C_i$ | coefficient in a series | $\overline{T}(s)$ | Laplace transform of $T(t)$ |
| $H(t)$ | Heaviside unit step function | $u(x,t)$ | dimensionless temperature |
| $K$ | thermal conductivity | $\beta$ | $= \pi^2$ |
| $\kappa$ | thermal diffusivity | $\Delta t$, $\Delta x$ | time increment and grid size, respectively |
| $M$ | integer | $\lambda_0$ | time lag |
| $T(t)$ | function of $t$ | $\tau_0$, $\tau_c$ | values of the dimensionless time lag |

$$\left\{1 + \lambda_0 \frac{\partial}{\partial \tau}\right\} \vec{q}(\vec{r},\tau) = -K\nabla\theta(\vec{r},\tau), \tag{5}$$

which has received a great deal of attention within the context of generalized thermoelasticity [3,4,6]. Combining Eq. (3) with Eq. (5), one may eliminate $\vec{q}$ and obtain the damped wave equation (DWE) [13–26]

$$\frac{\partial\theta(\vec{r},\tau)}{\partial\tau} + \lambda_0 \frac{\partial^2\theta(\vec{r},\tau)}{\partial\tau^2} = \kappa\nabla^2\theta(\vec{r},\tau). \tag{6}$$

In this study, we compare the difference between the solutions of the phase-lagging heat transport equation and the damped wave equation by investigating the solutions of a test problem. The solutions of the

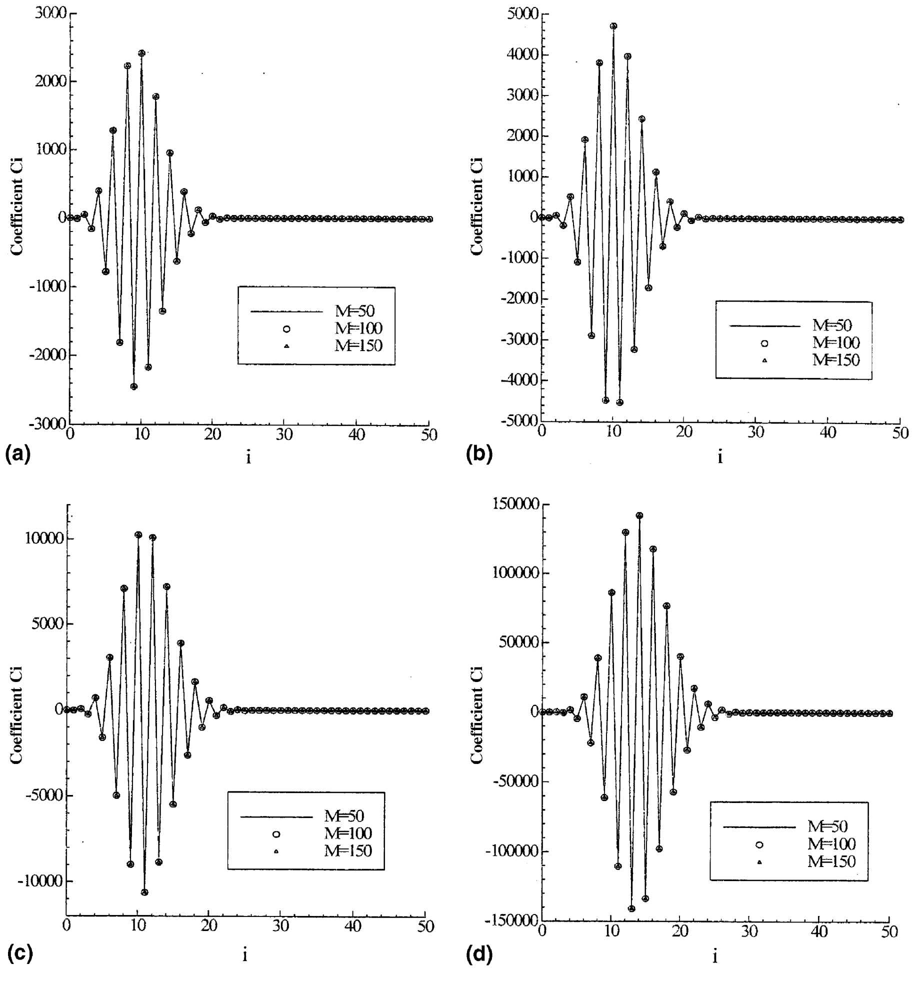


Fig. 1. Coefficient $C_i$ for (a) $\tau_0 = 0$, (b) $\tau_0 = 0.25\tau_c$, (c) $\tau_0 = 0.5\tau_c$ and (d) $\tau_0 = \tau_c$.

phase-lagging heat transport equation are obtained using the Laplace transform method and an approximate analytic method [27].

## 2. Problem formulations and solutions

Consider one-dimensional heat conduction in a thin, homogeneous, finite rod of constant cross-sectional area. Assuming that the rod has a constant thermal diffusivity, $\kappa > 0$, that occupies the open interval $(0, l)$ along the $\chi$-axis of a Cartesian coordinate system, and heat conduction within the rod is governed by the MC law, the mathematical model of this physical system consists of the following initial-boundary value problem (IBVP) [26]:

$$\frac{\partial\theta(\chi,\tau)}{\partial\tau} + \lambda_0\frac{\partial^2\theta(\chi,\tau)}{\partial\tau^2} = \kappa\frac{\partial^2\theta(\chi,\tau)}{\partial\chi^2}, \quad (\chi,\tau) \in (0,l)\times(0,\infty); \tag{7a}$$

$$\theta(0,\tau) = 0, \quad \theta(l,\tau) = 0, \quad \tau > 0; \tag{7b}$$

$$\theta(\chi,0) = \theta_0\sin[\pi\chi/l], \quad \partial\theta(\chi,0)/\partial\tau = 0, \quad \chi \in (0,l); \tag{7c}$$

where $\theta = \theta(\chi,\tau)$ denotes the temperature distribution in the rod. Here, we assume that the initial temperature of the rod is $\theta_0\sin[\pi\chi/l]$ and the temperature at both ends is maintained at 0. Furthermore, we assume that the lateral face of the rod is fully insulated, and $\partial\theta/\partial\tau = 0$ at $\tau = 0$. Introducing the following non-dimensional quantities:

$$u = \frac{\theta}{\theta_0}, \quad x = \frac{\chi}{l}, \quad t = \frac{\tau\kappa}{l^2}, \tag{8}$$

where $\theta_0 > 0$ is taken as a constant, IBVP (7) can be re-written in dimensionless form as

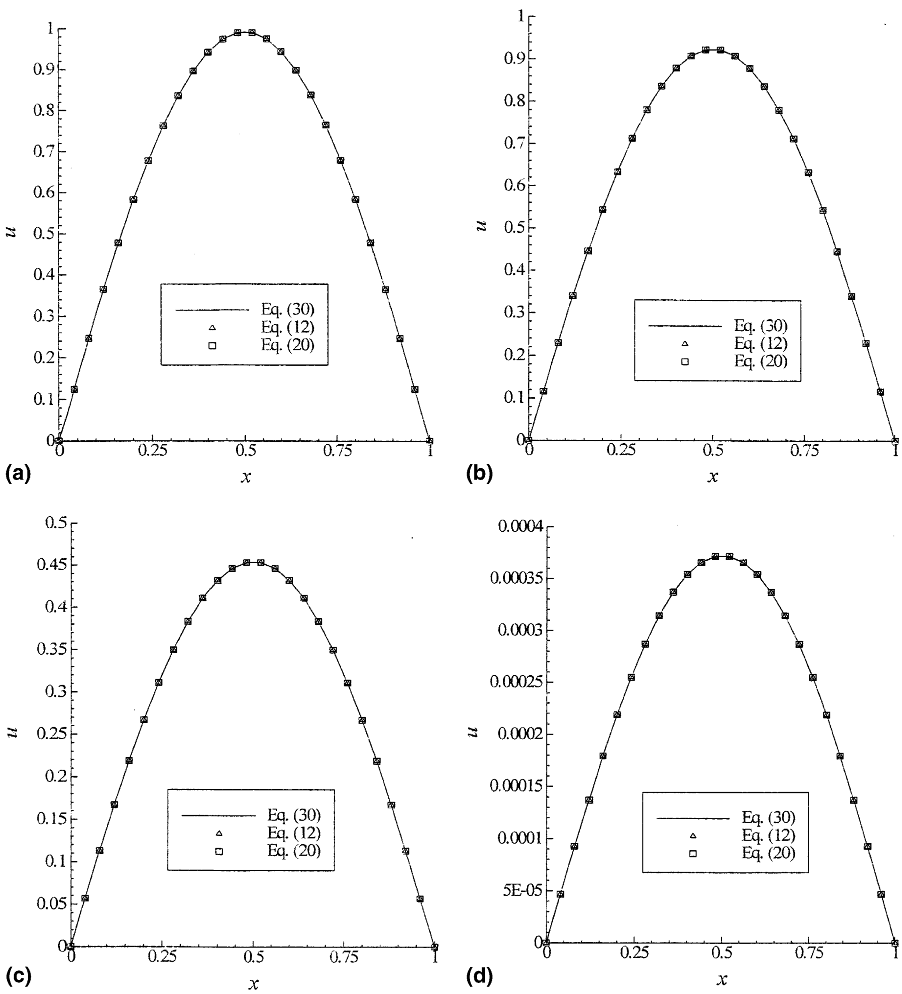


Fig. 2. $u$ vs. $x$ for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0$.

$$\frac{\partial u(x,t)}{\partial t} + \tau_0 \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2},$$
$$(x,t) \in (0,1) \times (0,\infty); \tag{9a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0; \tag{9b}$$

$$u(x,0) = \sin[\pi x], \quad \partial u(x,0)/\partial t = 0, \quad x \in (0,1); \tag{9c}$$

where the dimensionless lag time is given by

$$\tau_0 = \frac{\lambda_0 \kappa}{l^2}. \tag{10}$$

The exact solution to the above IBVP can be obtained using the separation of variables method and is given by [26]:

$$u(x,t) = \begin{cases} \exp[-t/(2\tau_0)] \sin[\pi x]\left\{\cosh[\omega t] + \frac{\sinh[\omega t]}{\sqrt{|\Delta|}}\right\}, \\ \quad \tau_0 < \tau_c, \\ \exp[-t/(2\tau_0)] \sin[\pi x]\left(1 + \frac{t}{2\tau_0}\right), \\ \quad \tau_0 = \tau_c, \\ \exp[-t/(2\tau_0)] \sin[\pi x]\left\{\cos[\omega t] + \frac{\sin[\omega t]}{\sqrt{|\Delta|}}\right\}, \\ \quad \tau_0 > \tau_c, \end{cases} \tag{11}$$

where $\tau_c \equiv (2\pi)^{-2}$ is a critical value of the thermal lag time, $\omega = (2\tau_0)^{-1}\sqrt{|\Delta|}$, and $\Delta = 1 - 4\pi^2\tau_0$. Here, however, we must reject the case $\tau_0 > \tau_c$ as it allows $u$ to assume negative values, in opposition to the fact that $u$ denotes an absolute quantity [26]. Furthermore, we note

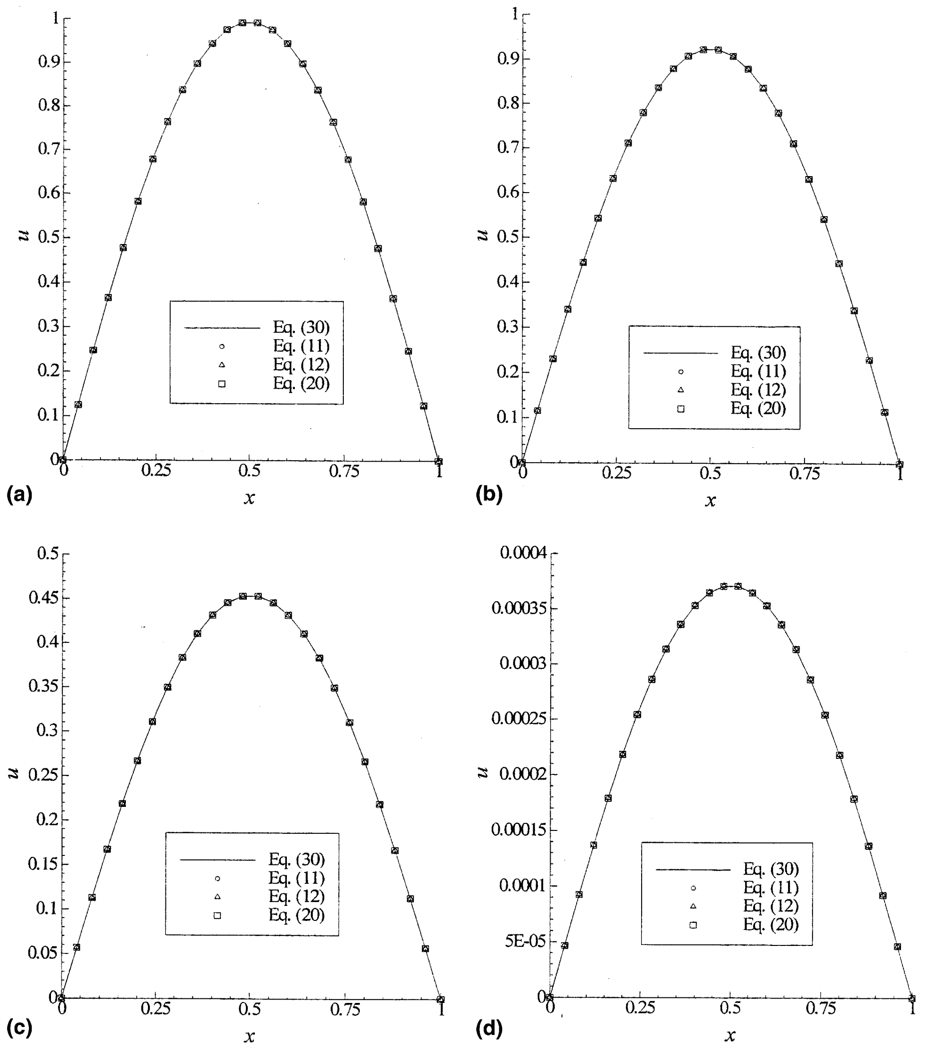


Fig. 3. $u$ vs. $x$ for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0.001\tau_c$.

that by letting $\tau_0 \to 0$ in Eq. (11), the classical Fourier-based solution is recovered, i.e.,

$$u(x,t) = e^{-\pi^2 t}\sin[\pi x]. \tag{12}$$

The corresponding (dimensionless) IBVP involving the phase-lagging model, Eq. (4), is given by

$$\frac{\partial u(x,t+\tau_0)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t) \in (0,1)\times(0,\infty); \tag{13a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0; \tag{13b}$$

$$u(x,t) = \sin[\pi x], \quad (x,t) \in (0,1)\times[-\tau_0, 0]; \tag{13c}$$

where the IC is now replaced by the specification of $u$ over an interval of time. In this section, the exact solution to IBVP (13) will be determined using the Laplace transform method. To this end, we assume a solution of the form

$$u(x,t) = T(t)\sin[\pi x]. \tag{14}$$

From this, it is not difficult to show that $T(t)$ satisfies the ordinary delay differential equation

$$T'(t+\tau_0) + \pi^2 T(t) = 0, \tag{15}$$

with the initial condition (IC)

$$T(t) = 1 \quad \text{when } t \in [-\tau_0, 0]. \tag{16}$$

Applying the Laplace transform, using the IC, and then solving the (algebraic) equation in the transform domain results in (see, e.g., [28])

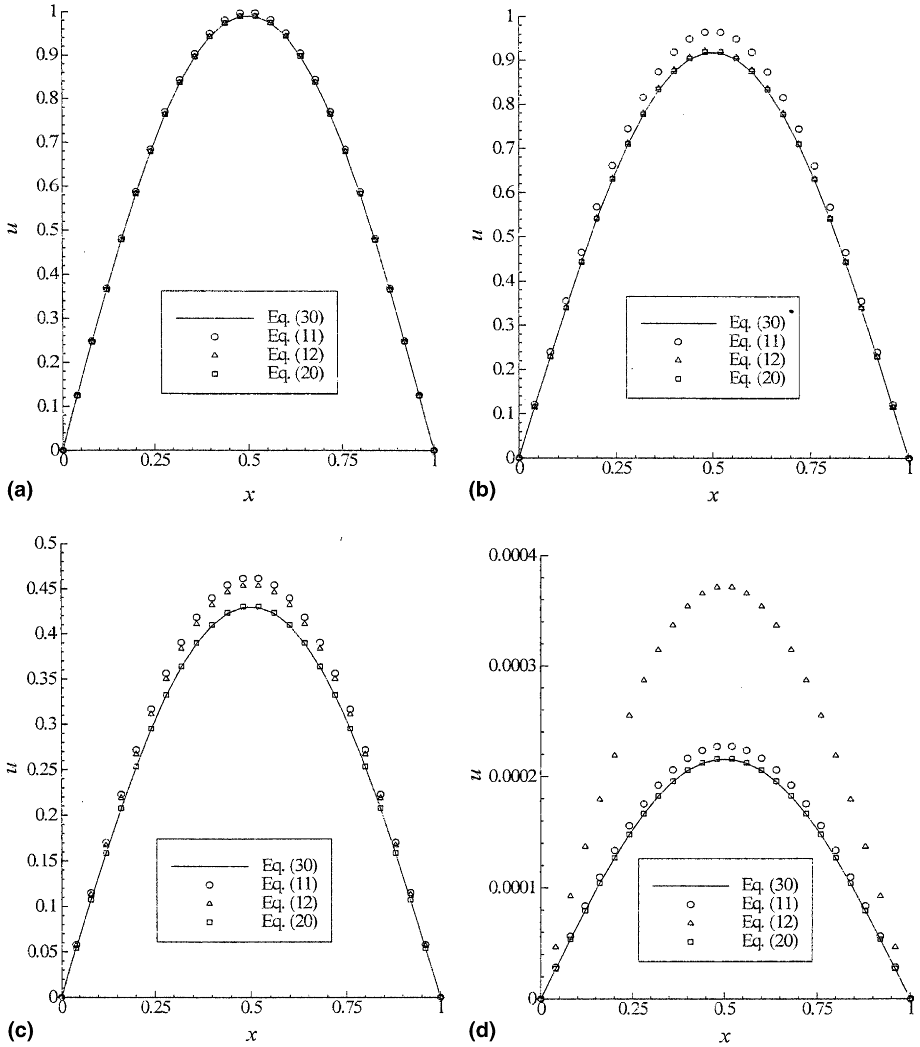


Fig. 4. $u$ vs. $x$ for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0.25\tau_c$.

$$\overline{T}(s) = \frac{1}{s}\left\{1 - \frac{\pi^2}{s + \pi^2 \exp[-s\tau_0]}\right\}, \tag{17}$$

where $\overline{T}(s)$ denotes the image of $T(t)$ is the Laplace transform domain and $s$ is the transform parameter. Next, expanding Eq. (17) in powers of $\frac{1}{s}$ yields

$$\overline{T}(s) = \frac{1}{s} + \sum_{m=0}^{\infty}\left(\frac{-\pi^2}{s}\right)^{m+1} \exp[-m(s\tau_0)]. \tag{18}$$

Finally, inverting term-by-term using a table of inverses along with the properties of the Laplace transform [28], we obtain, after some manipulation, the polynomial solution

$$T(t) = H(t)\left\{\sum_{m=0}^{\infty}(-\pi^2)^m H(t - \tau_0(m-1))\frac{(t - \tau_0(m-1))^m}{m!}\right\}, \tag{19}$$

where $H(\cdot)$ denotes the Heaviside unit step function. Hence, the exact solution to IBVP (13) is found to be

$$u(x,t) = H(t)\left\{\sum_{m=0}^{[t/\tau_0]+1}(-\tau_0\pi^2)^m \frac{(t/\tau_0 - (m-1))^m}{m!}\right\}\sin[\pi x], \tag{20}$$

where $[\cdot]$ denotes the greatest integer (or floor) function; i.e., $[p]$ denotes the greatest integer not larger than the real number $p$.

It should be pointed out that when $t \gg \tau_0$, the value of $[t/\tau_0]$ is very large, and thus our series solution may contain a large number of terms. Consequently, we now consider another solution method called the approximate analytical method [27], which is well-suited for the case $t \gg \tau_0$. To this end, we first rewrite Eq. (15) as follows:

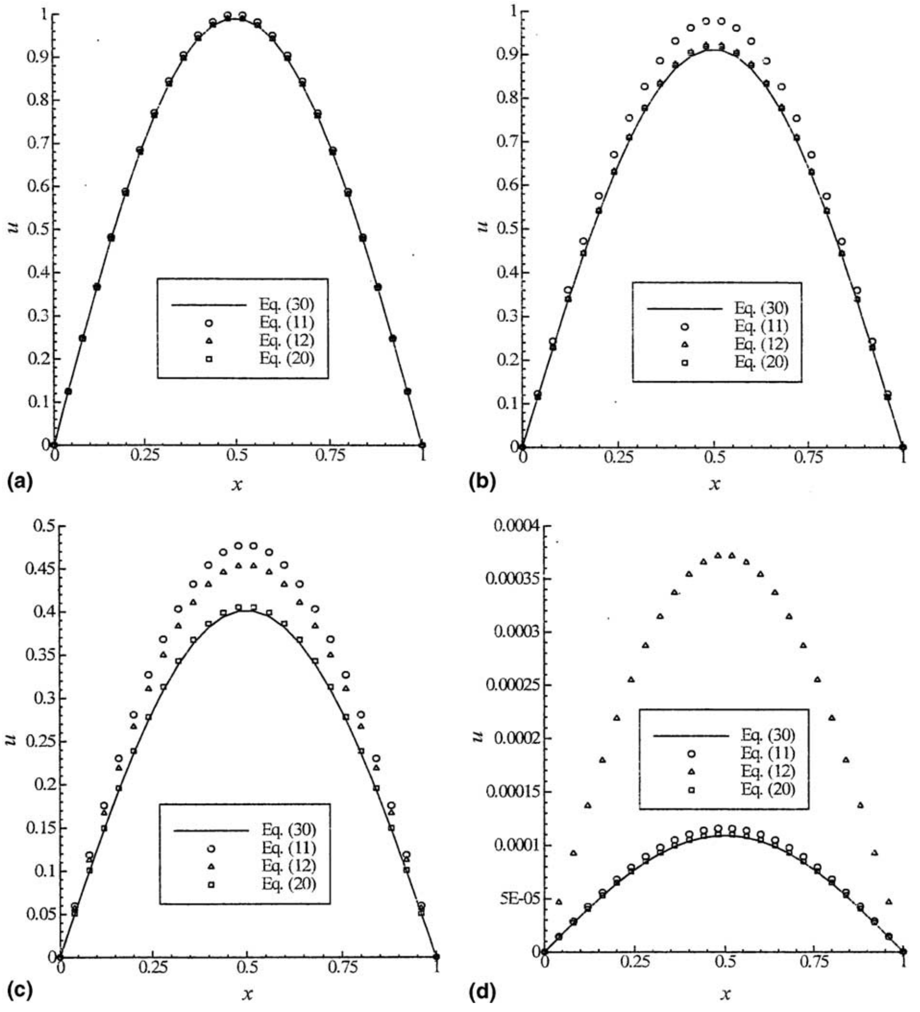


Fig. 5. $u$ vs. $x$ for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = 0.5\tau_c$.

$$T'(t+\tau_0) = \beta T(t), \tag{21}$$

where $\beta = \pi^2$. Now let

$$T(t) = \sum_{i=0}^{M} C_i t^i, \tag{22}$$

where $M$ is a large integer and $C_0 = T(0)$. Substituting Eq. (22) into Eq. (21) gives

$$\sum_{i=1}^{M} C_i i (t+\tau_0)^{i-1} = \beta \sum_{i=0}^{M} C_i t^i. \tag{23}$$

For $t = 0$ in Eq. (23), we obtain

$$\sum_{i=1}^{M} C_i i \tau_0^{i-1} = \beta C_0. \tag{24}$$

Differentiating Eq. (23) with respect to $t$, and setting $t = 0$, we obtain

$$\beta k! C_k = \sum_{i=k+1}^{M} C_i i(i-1)\cdots(i-k)\tau_0^{i-k-1},$$

$$k = 0, \ldots, M-1. \tag{25}$$

We now solve for $C_i$ ($i = 1, \ldots, M$) using Eq. (25). Letting $k = M-1$, gives

$$\beta C_{M-1}(M-1)! = C_M M!, \tag{26}$$

which can be rewritten as

$$C_M = C_{M-1}\left(\frac{a_{M-1}}{M}\right), \quad \text{where } a_{M-1} = \beta. \tag{27}$$

Letting $k = M-2, M-3, \ldots, 0$, we obtain expressions

$$a_{M-k} = \frac{\beta}{1 + \sum_{i=1}^{k-1} \frac{\tau_0^i}{i!} \prod_{j=1}^{i} a_{M-k+j}}, \quad k = 1, \ldots, M, \tag{28}$$

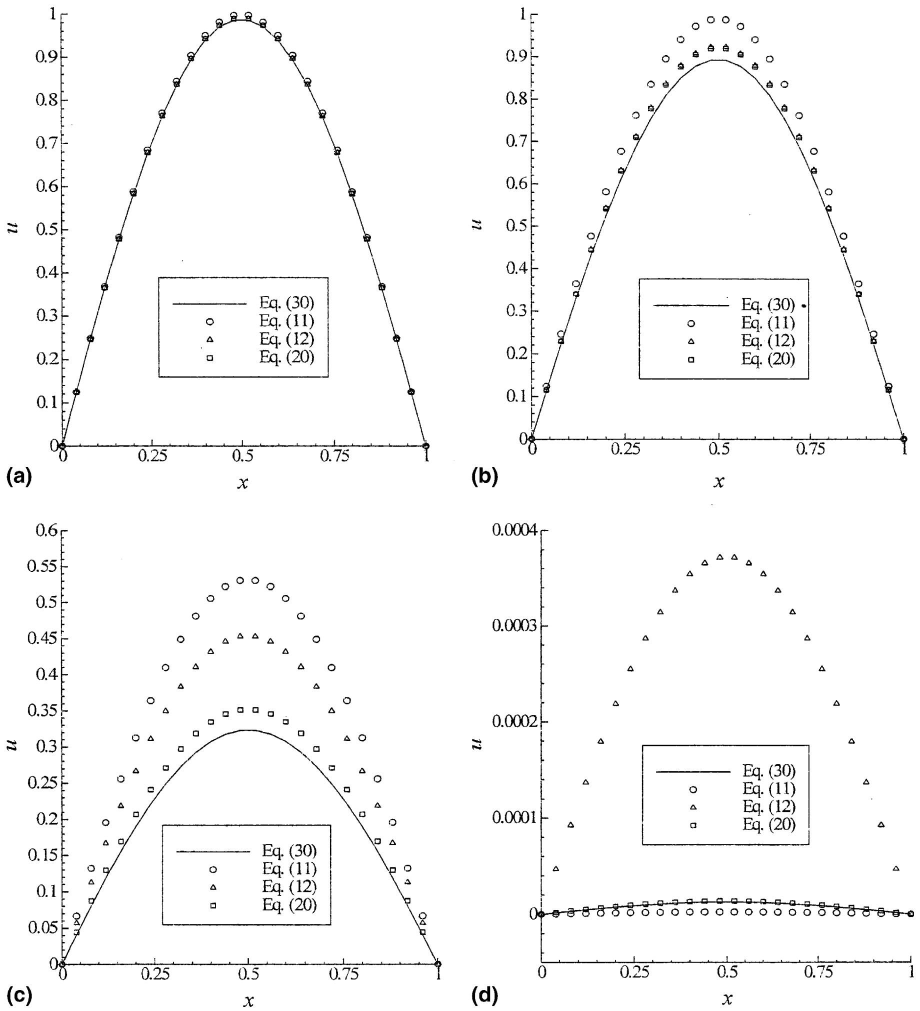


Fig. 6. $u$ vs. $x$ for (a) $t = 2(\Delta t)$, (b) $t = 20(\Delta t)$, (c) $t = 200(\Delta t)$, (d) $t = 2000(\Delta t)$; $\Delta x = 0.04$; $\Delta t = 0.0004$; and $\tau_0 = \tau_c$.

and

$$C_{M-k+1} = C_{M-k}\left(\frac{a_{M-k}}{M-k+1}\right),$$
$$\text{where } C_0 = T(0),\ \ k = 1,\ldots,M. \tag{29}$$

Once the $C_i$ ($i = 0,\ldots,M$) are obtained, $T(t)$ can be calculated by Eq. (22). Hence, we obtain the following approximate solution to the IBVP of Eq. (13):

$$u(x,t) = \sum_{i=0}^{M} C_i t^i \sin[\pi x]. \tag{30}$$

To determine how large the integer $M$ should be, we have, in Fig. 1, plotted the coefficients $C_i$ ($i = 0, 1, 2,\ldots,M$), where $M = 50, 100, 150$, for $\tau_0 = 0, 0.25\tau_c$, $0.5\tau_c$, $\tau_c$. It can be seen that for each value of $\tau_0$ considered, the coefficients do not change significantly as the value of $M$ is increased. Thus, we chose $M = 50$ in this study.

## 3. Numerical results and testing

We have computed and plotted Eq. (11), the exact solution to IBVP (9) involving the DWE, the exact and approximate solutions, Eqs. (20) and (30), respectively, to IBVP (13) involving the phase-lagging equation, and for comparison Eq. (12), the exact solution of the traditional heat conduction equation. In our computation, we chose the time increment, $\Delta t$, and the grid size, $\Delta x$, to be 0.0004 and 0.04, respectively.

In Figs. 2–6, we have plotted the temporal evolution of the temperature vs. $x$ profile for $\tau_0 = 0, 0.001\tau_c, 0.25\tau_c$, $0.5\tau_c$, $\tau_c$, the time-sequence consisting of the times of $2(\Delta t)$, $20(\Delta t)$, $200(\Delta t)$, and $2000(\Delta t)$. Clearly, Fig. 2 shows that these three solutions are the same when $\tau_0 = 0$, as expected. Fig. 3 shows that the four solutions overlap when $\tau_0 = 0.001\tau_c$. When $\tau_0 = 0.25\tau_c$, one may see from Fig. 4 that the solutions corresponding to Eqs. (20) and (30) are very close to each other, and the levels of both are lower than that of the solution given in Eq. (11). In particular, when $t = 2000(\Delta t)$, the level of the solution corresponding to Eq. (12) is much higher than the other three. Similar results can be seen in Figs. 5 and 6. Furthermore, one can see from Figs. 3–6 that by decreasing $\tau_0$, the solutions given in Eqs. (20) and (30) become "close" to the one corresponding to Eq. (11). This implies that when $\lambda_0 \to 0$, the DWE is a good approximation to the phase-lagging heat transport equation.

## 4. Conclusion

The differences between the solutions of the phase-lagging heat transport equation and the damped wave equation are compared by investigating the solutions of a test problem. Results show that the magnitude of the solution obtained by the phase-lagging heat transport equation is smaller than the one obtained by the DWE, and that the DWE is a good approximation of the phase-lagging heat transport equation when $\lambda_0$ is small. Our next task is to study the case where nonlinear source terms can appear.

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